Matching Condition on the Event Horizon and the Hologram Principle

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Abstract

It is shown that the event horizon for 4D black holes or $ds^2 = 0$ surfaces for multidimensional wormhole-like solutions reduces the amount of information necessary for defining all spacetime and hence satisfies the Hologram principle. This leads to the result that by matching two metrics on the $ds^2 = 0$ surfaces (event horizon for 4D black holes) we can match only the metric components but not its derivatives. For example, this allows us to obtain a composite wormhole by inserting a 5D wormhole-like flux tube between two Reissner-Nordström black holes and matching it at the event horizon. Using the Hologram principle the entropy of the black hole from algorithm theory is obtained.

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I. INTRODUCTION

Matching two metrics which are solutions of the Einstein equations leads to the fact that a surface stress-energy appears on the matching surface. This result is a consequence of the Einstein equations. The detailed explanation for this can be found, for example, in Ref. [1]. The cause of this is evident: the Riemann tensor has second derivatives of the metric that brings a δ-function in the left side of the Einstein equations, hence on the right side there should be a δ-like surface stress-energy.

But the Hologram principle proposed in the Ref's [2], [3], [4] indicates that there is a surface on which one essentially cuts down the number of degrees of freedom. It can be assumed that matching two metrics on this surface can substantially change the matching procedure on this surface. For this purpose we examine the Lorentzian invariant surface on which $ds^2 = 0$ [1].

Further we consider several solutions in 4D and vacuum multidimensional (MD) gravity: two solutions are the Reissner-Nordström and Yang-Mills black holes (BH) and two solutions are wormhole-like (WH) solutions that in some sense are dual to the above-mentioned BHs [2].

To begin with we repeat the definition of the Hologram principle given in [4]: "... a full description of nature requires only a two dimensional lattice at the spatial boundaries of the

[1] often this is an event horizon (EH) but for some wormhole-like, nonasymptotically flat, multidimensional solutions it is not so.

[2] This duality means that the static region of the 4D BHs are disposed at $r \geq r_g$ but the static region of the MD wormhole-like solutions at $|r| \leq r_g$.

world ...“. Our aim is to show that this principle works on the $ds^2 = 0$ surface.

II. 4D CASE

A. Event horizon as the Hologram surface for the Reissner-Nordström BH

The metric in this case is:

$$ds^2 = \Delta(r)dt^2 - \frac{dr^2}{\Delta(r)} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2,\right) \tag{1}$$

the electromagnetic potential is:

$$A_\mu = u = \{\omega(r), 0, 0, 0\}. \tag{2}$$

The Einstein - Maxwell equations can be written as:

$$-\frac{\Delta'}{r} + \frac{1-\Delta}{r^2} = \frac{\kappa}{2}\omega'^2, \tag{3}$$

$$-\frac{\Delta''}{2} - \frac{\Delta'}{r} = -\frac{\kappa}{2}\omega'^2, \tag{4}$$

$$\omega' = \frac{q}{r^2}. \tag{5}$$

It is easy to prove that Eq. (4) is a consequence of Eqs. (3) and (5) For the Reissner - Nordström BH the event horizon (EH) is defined by $\Delta(r_g) = 0$, where r_g is the radius of the EH. Hence in this case we see that on the EH:

$$\Delta'_g = \frac{1}{r_g} - \frac{\kappa}{2}r_g{\omega'_g}^2, \tag{6}$$

here (g) means that the corresponding value is taken on the EH. Thus, Eq. (3), which is the Einstein equation, is a first order differential equation in all spacetime ($r \geq r_g$). The condition in Eq. (6) indicates that the derivative of the metric on the EH is expressed through the metric value on the EH. This means that the Hologram principle works here and is connected with the presence of the EH. In passing we remark that this allows us to calculate an entropy of the BH from the algorithmical point of view [5] without any quantum mechanical calculations. This calculation will be done in IV B.

B. Event horizon as the Hologram surface for SU(2) Yang-Mills BH

Here we use the following metric:

$$ds^2 = e^{2\nu(r)}\Delta(r)dt^2 - \frac{dr^2}{\Delta(r)} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{7}$$

For the SU(2) Yang-Mills gauge field we choose the following monopole-like ansatz as in [6]:

$$A^a_\mu = \frac{\epsilon^a_{ij}x^j}{r^2}\left(1 - f(r)\right), \tag{8}$$

$$A^a_t = \frac{x^a}{r^2}v(r), \tag{9}$$

here $a = 1, 2, 3$ is the inner index; $i = 1, 2, 3$ is the space index. For simplicity we consider the $v = 0$ case. Thus, we have the following Einstein - Yang - Mills system of equations :

$$-\frac{\Delta'}{r} + \frac{1-\Delta}{r^2} = \frac{\kappa}{r^2}\left[\Delta f'^2 + \frac{1}{2r^2}\left(f^2 - 1\right)^2\right], \tag{10}$$

$$\nu' = \frac{\kappa}{r}f'^2, \tag{11}$$

$$R^\theta_\theta - \frac{1}{2}R = \kappa T^\theta_\theta, \tag{12}$$

$$\Delta f'' + \Delta f'\nu' + f'\Delta' = \frac{f}{r^2}\left(f^2 - 1\right). \tag{13}$$

On account of the Bianchi identity Eq. (12) is a consequence of the other equations. From (11) we have:

$$\nu(r) = \kappa\int_r^\infty \frac{f'^2}{r}dr, \tag{14}$$

here we choose the time so that $\nu_{r\to\infty}=0$. With this the value on the EH is

$$\nu_g = \kappa \int_{r_g}^{\infty} \frac{f'^2}{r} dr. \tag{15}$$

The presence of the EH means that $\Delta(r_g)=0$, hence close to the EH:

$$\Delta = \Delta_1 x + \Delta_2 \frac{x^2}{2} + \cdots, \tag{16}$$

$$f = f_0 + f_1 x + f_2 \frac{x^2}{2} + \cdots, \tag{17}$$

here $x = r - r_g$. Then from Einstein - Yang - Mills equations we have:

$$f_2 = -\frac{\kappa}{r_g} f_1^3, \tag{18}$$

$$f_1 = \frac{f_0}{r_g^2 \Delta_1}\left(f_0^2 - 1\right), \tag{19}$$

$$\Delta_1 = \frac{1}{r_g} - \frac{\kappa}{2r_g^3}\left(f_0^2 - 1\right)^2. \tag{20}$$

Thus, we have only one physically significant parameter - $f_0 = f(r_g)$ [3]. Again we have the Hologram principle on the EH. Furthermore, the Yang - Mills equations satisfy the Hologram principle since the first derivative of $f(r)$ on the EH is expressed through the value of f_0 on EH, as can be seen from equation (19) .

III. MULTIDIMENSIONAL WORMHOLE-LIKE CASES

A. $ds^2 = 0$ surface as Hologram surface for 5D WH-like solution

Let us consider the 5D WH-like metric:

$$ds^2 = \frac{1}{\Delta(r)} dt^2 - R_0^2 \Delta(r)\Big[d\chi + \omega(r) dt\Big]^2 - dr^2 - a(r)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{21}$$

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[3] r_g can be excluded by making all the magnitudes dimensionless ($r^* = r/r_g$; $\kappa' = \kappa/r_g^2$).

here χ is the 5^{th} coordinate, r, θ and φ are the ordinary spherically symmetric coordinates, R_0 is some constant. The 5D Einstein equations are:

$$\Delta\Delta'' - \Delta'^2 + \Delta\Delta'\frac{a'}{a} + R_0^2\Delta^4\omega'^2 = 0, \tag{22}$$

$$\left(a\Delta^2\omega'\right)' = 0, \tag{23}$$

$$a'' = 2. \tag{24}$$

The solution is [7]:

$$a = r^2 + r_0^2 \tag{25}$$

$$\Delta = \frac{q}{2r_0}\frac{r_0^2 - r^2}{r_0^2 + r^2}, \tag{26}$$

$$\omega = \frac{4r_0^2}{q}\frac{r}{r_0^2 - r^2}. \tag{27}$$

here q and r_0 are some constants. It is easy to show that $G_{tt}(r_0) = \Delta^{-1}(r_0) - R_0^2\Delta(r_0)\omega^2(r_0) = 0$ and $ds^2 = 0$ on the surface $r = r_0$. In this sense the $r = r_0$ surface is analogous to the EH. On the $ds^2 = 0$ surface $\Delta(r_0) = 0$, and this leads to the result that from Eq. (22) we have:

$$\Delta'_0 = \pm\frac{q}{a_0} = \pm\frac{q}{2r_0^2}. \tag{28}$$

The signs ($\pm$) refer respectively to ($r = \mp r_0$) which are the $ds^2 = 0$ surfaces. This also indicates that the $ds^2 = 0$ surfaces work here according to the Hologram principle.

B. $ds^2 = 0$ surface as Hologram surface for 7D WH-like solution

Here we work with gravity on the principal bundle as in Ref. [8], i.e. the base of the bundle is an ordinary 4D Einstein spacetime and the fibre of the bundle is the SU(2) gauge group. In our case we have gravity on the SU(2) principal bundle with an SU(2) structural group (simultaneously it is the extra coordinates). This group is the space of the extra dimensions. Thus, the dimension of our MD gravity is seven.

The gravity equations are:

$$R_{a\mu} = 0, \tag{29}$$

$$R^a_a = R^4_4 + R^5_5 + R^6_6 = 0, \tag{30}$$

here $A = 0, 1, 2, \ldots, 6$ is an MD index on the total space of the bundle, $\mu = 0, 1, 2, 3$ is the index on the base of the bundle, $a = 4, 5, 6$ is the index on the fibre of the bundle. For MD gravity on the principal bundle we have the following theorem [9,10]:

Let G be the group fibre of the principal bundle. Then there is a one-to-one correspondence between the G-invariant metrics on the total space $\mathcal{X}$ and the triples $(g_{\mu\nu}, A^a_\mu, h\gamma_{ab})$. Here $g_{\mu\nu}$ is Einstein's pseudo - Riemannian metric on the base; A^a_μ is the gauge field of the group G (the nondiagonal components of the multidimensional metric); $h\gamma_{ab}$ is the symmetric metric on the fibre.

In accord with this theorem the 7D metric has the following form:

$$ds^2 = \frac{\Sigma^2(r)}{u^3(r)}dt^2 - R_0^2 u(r)\left(\sigma^a + A^a_\mu dx^\mu\right)^2 - dr^2 - a(r)\left(d\theta^2 + \sin\theta d\varphi^2\right), \tag{31}$$

here A^a_μ is the above-mentioned SU(2) gauge field, the one-forms σ^a on the SU(2) group can be written as follows:

$$\sigma^1 = \frac{1}{2}(\sin\alpha d\beta - \sin\beta\cos\alpha d\gamma), \tag{32}$$

$$\sigma^2 = -\frac{1}{2}(\cos\alpha d\beta + \sin\beta\sin\alpha d\gamma), \tag{33}$$

$$\sigma^3 = \frac{1}{2}(d\alpha + \cos\beta d\gamma), \tag{34}$$

here we have introduced the Euler angles α, β, γ on the fibre ($SU(2)$ group) with $0 \le \beta \le \pi, 0 \le \gamma \le 2\pi, 0 \le \alpha \le 4\pi$.

The ansatz for the gauge potential A^a_μ we take of the monopole form (as in section II B). For simplicity we examine only the $f(r) = 0$ case. This gives the following vacuum gravity equations:

$$v'' - \frac{\Sigma' v'}{\Sigma} + 4\frac{u'v'}{u} + \frac{a'v'}{a} = 0, \tag{35}$$

$$\frac{a''}{a} + \frac{a'\Sigma'}{a\Sigma} - \frac{2}{a} + \frac{R_0^2 u}{4a^2} = 0, \tag{36}$$

$$\frac{u''}{u} + \frac{\Sigma' u'}{\Sigma u} - \frac{u'^2}{u^2} + \frac{a'u'}{au} - \frac{4}{R_0^2 u} - \frac{1}{12}\frac{R_0^2 u}{a^2} + \frac{1}{3}\frac{R_0^2 u^4}{\Sigma^2} v'^2 = 0, \tag{37}$$

$$\frac{\Sigma''}{\Sigma} + \frac{a'\Sigma'}{a\Sigma} - \frac{6}{R_0^2 u} - \frac{R_0^2 u}{8a^2} = 0. \tag{38}$$

In Ref. [11] an approximative solution was found. Here we are interested only in what happens close to the $ds^2 = 0$ surface. In order that the $ds^2 = 0$ surface exist it is necessary that the following condition be satisfied:

$$G_{tt} = \frac{\Sigma_0^2}{u_0^3} - R_0^2 u_0 v_0^2 = 0, \tag{39}$$

here the index (0) means that the magnitudes are taken on the $r = r_0$ surface. We assume that in this region there is the following behaviour:

$$u(r) = u_0 \left(1 - \frac{r}{r_0}\right)^{1/2} + \cdots, \tag{40}$$

$$\Sigma(r) = \Sigma_0 + \Sigma_1 \left(1 - \frac{r}{r_0}\right)^{3/2} + \cdots, \tag{41}$$

$$v(r) = \frac{q\Sigma_0}{a_0 u_0^4} \frac{1}{1 - \frac{r}{r_0}} + \cdots. \tag{42}$$

This leads to the following result:

$$u_0^2 = \sqrt{\frac{2}{3}} \frac{|q| r_0}{a_0}, \tag{43}$$

$$R_0 = \sqrt{\frac{2}{3}} u_0^2 r_0, \tag{44}$$

$$\frac{\Sigma_1}{\Sigma_0} = \frac{12}{u_0^5}. \tag{45}$$

Here we also see the Hologram principle: u_0 and $\Sigma'(r_0) = \Sigma_1$ are not independent initial data; they are defined by the dimensionless quantities q/r_0 and a_0/r_0^2.

IV. DISCUSSION

Thus, we see that at least for the static, spherically symmetric solution in 4D and vacuum MD gravity the Hologram principle follows from the presence of the $ds^2 = 0$ surface (event horizon for 4D gravity). For researchers working with 4D Einstein - Yang - Mills black holes this is well known: condition (19) is necessary for numerical calculations (see, for example, [6]).

These results allow us to infer that (at least for static, spherically symmetric cases) on the $ds^2 = 0$ surfaces the Hologram principle changes and simplifies the matching conditions as a consequence of a reduction in the physical degrees of freedom. Roughly speaking, close to this surface Einstein's second order differential equations are reduced to first order equations [4]. In this case it is evident that: *matching two metrics on the $ds^2 = 0$ surface does not lead to $\delta-$functions and hence the appearance of an additional surface stress-energy.*

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[4]for the 4D spherically symmetric case it is an exact result.

A. Composite WH with 5D WH-like solution and two Reissner-Nordström black holes

As a further example, 5D WH-like solutions can be matched to two Reissner-Nordström black holes on the two $ds^2 = 0$ surfaces [12]. This can be done since ordinary 4D electro-gravity can be considered as 5D vacuum gravity in the initial Kaluza sense [5], and on the EH we sew fibre to fibre and base to base the Reissner - Nordström and 5D WH-like solutions. In this case we have to match on the $ds^2 = 0$ surfaces (EHs for the observer at infinity) only the following magnitudes:

- The area of the $ds^2 = 0$ surfaces of the 5D WH-like solution with the area of the EH of the Reissner - Nordström BH:

$$4\pi a_0 = 4\pi r_g^2 \tag{46}$$

here the left side of this equation is 5D and the right side is 4D.

- Let us compare the $R_{15} = 0$ 5D equation:

$$\left(4\pi a\omega' e^{-2\nu}\right)' = 0 \tag{47}$$

with the Maxwell equation:

$$\left(4\pi r^2 E\right)' = 0. \tag{48}$$

In both cases $4\pi a$ and $4\pi r^2$ are the areas of the 2-sphere and Eqs. (47) and (48) tell us that a flux of electrical field is preserved. Hence we can conclude that $\omega' e^{-2\nu}$ is the

[5] when $G_{55} = 1$ and the Lagrangian does not vary with respect to G_{55}.

"5D electrical" field $E_5 = \omega' e^{-2\nu}$, and for the 4D case we have the ordinary definition for electrical field. Hence on the $ds^2 = 0$ matching surface we will match the flux tubes of both electrical fields:

$$4\pi a_0 \omega_0' e^{-2\nu_0} = 4\pi r_g^2 E_g \tag{49}$$

here (0) means that the corresponding 5D magnitude is evaluated on the $ds^2 = 0$ surface, and (g) means that the 4D magnitude is taken on the EH.

- We do not match G_{rr} and g_{rr} since these components of the 5D and 4D metrics are arbitrary: they depend only on the choice of the radial coordinate.

In fact we see that we have only two matching conditions : the Reissner - Nordström BH is characterized only by two physical quantities - electrical charge Q and mass m. The 5D WH-like solution (25) - (27) is characterized only by two physical quantities - q and r_0. Remarkablly for these 4D and 5D physical quantities we have only two matching conditions (46) and (49) as a consequence of the Hologram principle.

B. Hologram principle and algorithmical complexity

It is interesting that the reduction of the order of Einstein's differential equations near the EH and the Hologram principle allows us to calculate the entropy of the BH without any quantum calculations. In short we repeat the result of Ref. [5]. In the 1960's Kolmogorov postulated that algorithm theory allows one to define a notion of probability for a single object. His idea is very simple: the probability is connected with the complexity of this object, "chance" = "complexity". The more complicated (longer) an algorithm is which describes this object [6] the smaller a probability it has. Of course the question is then: what is the length of the algorithm ? It was found that such an invariant, well defined definition can be given [13]:

The algorithmic complexity $\mathcal{K}(x \mid y)$ of the object x for the defined object y is the minimal length of the "program", P, which is written as a sequence of the zeros and ones which allows one to construct x when y is given:

$$\mathcal{K}(x \mid y) = \min_{A(P,y)=x} l(P) \tag{50}$$

where $l(P)$ is the length of the program P; $A(P,y)$ is the algorithm of the calculated object x, using the program P, when the object y is given. Then we can define the algorithmical complexity of the BH, and the logarithm of this gives us the entropy of the BH.

We write the initial equations for describing the Schwarzschild BH. The metric is:

$$ds^2 = dt^2 - e^{\lambda(t,R)}dR^2 - r^2(t,R)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{51}$$

here t is time, R is radius, θ and ϕ are polar angles. In this case Einstein's equations are:

$$-e^{-\lambda}r'^2 + 2r\ddot{r} + \dot{r}^2 + 1 = 0, \tag{52}$$

$$-\frac{e^{-\lambda}}{r}\left(2r'' - r'\lambda'\right) + \frac{\dot{r}\dot{\lambda}}{t} + \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} + \frac{2\ddot{r}}{r} = 0, \tag{53}$$

$$-\frac{e-\lambda}{r^2}\left(2rr'' + r'^2 - rr'\lambda'\right) + \frac{1}{r^2}\left(r\dot{a}\dot{\lambda} + \dot{a}^2 + 1\right) = 0, \tag{54}$$

$$2\dot{r}' - \dot{\lambda}r' = 0, \tag{55}$$

where $(')$ and $(\dot{\ })$ mean respectively the derivatives on t and r. The $\binom{0}{0}$ Einstein's equation for the initial data is:

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[6] such an algorithm can be, for example, the field equations describing field distributions in spacetime.

$$-\frac{e^{-\lambda}}{r^2}\left(2rr'' + r'^2 - rr'\lambda'\right) + \frac{1}{r^2}\left(r\dot{a}\dot{\lambda} + \dot{a}^2 + 1\right) = 0. \tag{56}$$

The Cauchy hypersurface for defining all Schwarzschild - Kruskal spacetime is $t = 0$. The "quantity" of the initial data can be essentially reduced according to the Hologram principle: at $t = 0$ the first time derivative of all metric components is equal to zero. Therefore we have the following equation for the initial data:

$$2rr'' + r'^2 - rr'\lambda' - e^{\lambda} = 0. \tag{57}$$

We know that the hypersurface $t = 0$ is a WH connecting two asymptotically flat, causally nonconnected regions. This WH is symmetrical with regard to $r = r_g$, therefore the initial data for this equation are:

$$r'(R = 0, t = 0) = 0, \tag{58}$$

$$r(R = 0, t = 0) = r_g, \tag{59}$$

where r_g is the radius at the event horizon. The conditions (58) and (59) are necessary for the existence of this WH, and this is also a consequence of the Hologram principle (reducing the information describing the BH). Thus, for describing all Schwarzschild - Kruskal spacetime we have to have the algorithm (52) - (55) and the initial data (59). Therefore the algorithmical complexity of the Schwarzschild BH $\mathcal{K}$ is defined by the following expression:

$$\mathcal{K} \approx L_{initial}\left(\frac{r_g}{r_{Pl}}\right)^2 + L_{Einstein\ equations}, \tag{60}$$

where $L_{initial}$ is the length of the algorithm (program) defining the dimensionless number r_g^2/r_{Pl}^2, made up on some universal machine; $L_{Einstein\ equations}$ is the length of the algorithm (program) for solving the Einstein equations (52) - (55) on the some universal machine, (*e.g.* a Turing machine).

V. CONCLUSION

Finally, we can postulate that the event horizon plays an exceptional role in Nature: *it is the surface on which the Hologram principle is realized*. We have shown that in this case the EH can *divide* the regions in our Universe with splitting off and nonsplitting off of the extra dimensions in the above mentioned sense as a consequence of the realization of the Hologram principle.

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